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Nicolas Borie

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COMBINATORICS OF SIMPLE MARKED MESH PATTERNS IN 132-AVOIDING PERMUTATIONS

NICOLAS BORIE

ABSTRACT. We present some combinatorial interpretations for coefficients appearing in series partitioning the permutations avoiding 132 along marked mesh patterns. We study patterns in which only one parameter is non zero the combinatorial family in bijection with 132-avoiding permutations and also preserving the statistic counted by the marked mesh pattern. Following the works of Kitaev, Remmel and Tiefenbruck, we present alternative proofs for two quadrants and give a combinatorial interpretation for the last one exhibiting a new bijection between 132-avoiding permutations and non-decreasing parking functions.

1. INTRODUCTION

Mesh patterns were introduced by Brändén and Claesson [BC11] to provide explicit expansions for certain permutation statistics as (possibly infinite) linear combinations of (classical) permutation patterns. This notion was further studied by Úlfarsson [Úlf11] and Kitaev, Remmel and Tiefenbruck in some series of papers refining conditions on permutations and patterns. The present paper focuses on what the trio Kitaev, Remmel and Tiefenbruck call *simple marked pattern in 132-avoiding permutations* [KRT12].

Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation written in one-line notation. We consider the graph $G(\sigma)$ of σ to be the set of points $\{(i, \sigma_i) : 1 \leq i \leq n\}$.

Now, if we draw a coordinate system centered at the point (i, σ_i) , we are interested in the points that lie in the four quadrants I, II, III, IV of that coordinate system as represented on the right of Figure 1. For any a, b, c, d , four non negative integers, we say that σ_i matches the simple marked mesh pattern $MMP(a, b, c, d)$ in σ if, in the coordinate system centered at (i, σ_i) , $G(\sigma)$ has at least a, b, c and d points in the respective quadrants I, II, III and IV.

For $\sigma = 768945213$, then $\sigma_6 = 5$ matches the simple marked mesh pattern $MMP(0, 3, 1, 1)$, since relative to the coordinate system with origin $(6, 5)$, $G(\sigma)$ has respectively 0, 4, 1 and 3 points in Quadrants I, II, III and IV (see Figure 1). If a coordinate in $MMP(a, b, c, d)$ is zero, then there is no condition on the number of points in the corresponding quadrant. We let $mmp(a, b, c, d)(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a, b, c, d)$ in σ .

Given a sequence $w = w_1 \dots w_n$ of distinct integers, let $pack(w)$ be the permutation obtained by replacing the i -th largest integer that appears in w by i . For example, if $w = 2754$, then $pack(w) = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group \mathfrak{S}_j , we say that the pattern τ occurs in a permutation $\sigma \in \mathfrak{S}_n$ if there exist $1 \leq i_1 \leq \dots \leq i_j \leq n$ such that $pack(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$. We say that a permutation σ avoids the pattern τ if τ does not occur in σ . We will denote by $\mathfrak{S}_n(\tau)$ the set of permutations in \mathfrak{S}_n avoiding τ . This paper presents some combinatorial results concerning the generating functions

$$(1) \quad Q_{132}^{(a,b,c,d)}(t, x) := 1 + \sum_{n \geq 1} t^n Q_{n,132}^{(a,b,c,d)}(x)$$

Key words and phrases. permutation statistics, marked mesh pattern, distribution, Catalan numbers, Narayana numbers.

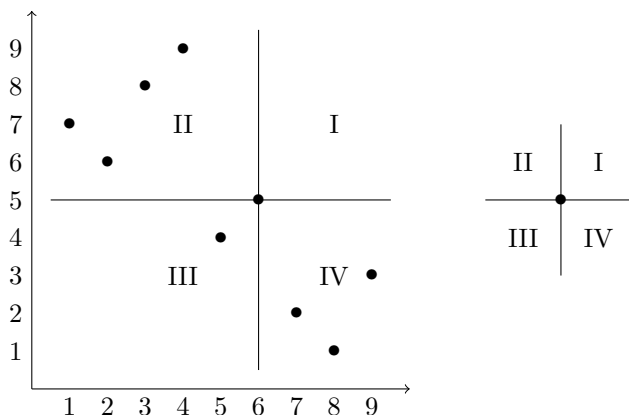


FIGURE 1. The graph of $\sigma = 768945213$ with coordinate system at position 6.

where for any $a, b, c, d \in \mathbb{N}$,

$$(2) \quad Q_{n,132}^{(a,b,c,d)}(x) := \sum_{\sigma \in \mathfrak{S}_n(132)} x^{mmp(a,b,c,d)(\sigma)}.$$

More precisely, we give a combinatorial interpretation of each coefficient of these series when exactly one value among a, b, c, d is non zero. Note that, as explained in [KRT12], $Q_{132}^{(0,0,0,k)}(t, x) = Q_{132}^{(0,k,0,0)}(t, x)$, so that we shall not consider the fourth quadrant.

2. QUADRANT I: THE SERIES $Q_{132}^{(\ell,0,0,0)}$

The marked mesh pattern $MMP(\ell, 0, 0, 0)$ splits the 132-avoiding permutations according to the number of values having at least ℓ greater values on their right.

We recall that a Dyck path with $2n$ steps is a path on the square lattice with steps $(1, 1)$ or $(1, -1)$ from $(0, 0)$ to $(2n, 0)$ that never falls below the x-axis.

Theorem 2.1. $Q_{132}^{(\ell,0,0,0)}|_{t^n x^k}$ counts the number of Dyck paths with $2n$ steps having exactly k steps $(1, -1)$ ending at height greater than or equal to ℓ .

Sketch of proof : The key is the Krattenthaler's bijection [Kra01] between 132-avoiding permutations and Dyck paths.

As the number of increasing steps from height at least 1 in Dyck paths are counted by the Catalan triangle, we deduce the following corollary.

Corollary 2.2. $Q_{132}^{(1,0,0,0)}|_{t^n x^k}$ is equal to the value $C_{n,k}$ of the Catalan triangle.

To illustrate Theorem 2.1, here is an example with $\ell = 2$ and $n = 4$. We have $Q_{4,132}^{(2,0,0,0)}(x) = 8 + 4x + 2x^2$ and Figure 2 shows the fourteen Dyck paths of length 8 with marked decreasing steps ending at height greater than or equal to 2. We have eight Dyck paths under the horizontal line at height 2, four paths containing a single decreasing step ending at height 2, and two last paths containing 2 such decreasing steps.

3. QUADRANT II: THE SERIES $Q_{132}^{(0,\ell,0,0)}$

Quadrant II is about classifying 132-avoiding permutations according to the number of values having ℓ greater values on their left. We will see that this statistic can be read on first values only.

Let n be a positive integer, we will call non-decreasing parking functions of length n the set of all non-decreasing functions f from $1, 2, \dots, n$ into $1, 2, \dots, n$ such that for all $1 \leq i \leq n$: $f(i) \leq i$. It is well-known that these are enumerated by Catalan numbers.

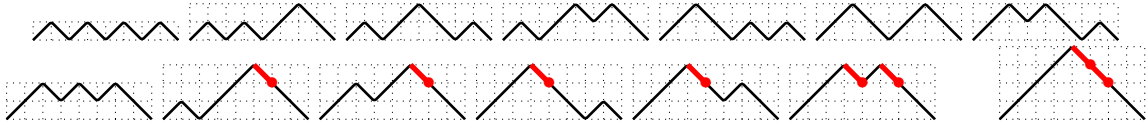


FIGURE 2. Decreasing steps ending at height greater than or equal to 2 in Dyck paths of length 8.

3.1. The general result.

Proposition 3.1. *Let σ be a 132-avoiding permutation. The statistic $mmp(0, 1, 0, 0)$ only depends on the first value of σ . More precisely, $mmp(0, 1, 0, 0)(\sigma) = \sigma_1 - 1$.*

This first proposition is the key to understanding the combinatorics attached to quadrant II. We will also understand it as follows: if a position i matches $MMP(0, 1, 0, 0)$ in σ , thus $\sigma_1 \geq \sigma_i$.

Theorem 3.2. $Q_{132}^{(0, \ell, 0, 0)}|_{t^n x^k}$ is equal to the number of 132-avoiding permutations in \mathfrak{S}_n such that the packed of their suffix of length $n + 1 - \ell$ begins with value $k + 1$.

For example, with $\ell = 2$ and $n = 4$, we have $Q_{4,132}^{(0, 2, 0, 0)}(x) = 4 + 6x + 4x^2$. The following table displays the fourteen permutations of length 14 avoiding 132, their suffix of length $4 + 1 - 2 = 3$, the packed of the suffix and the statistic $mmp(0, 2, 0, 0)$. The permutation are enumerated by lexicographic order, however the reader can check that four of them contain no value having two greater values on their left, six contain a single such value and the last four contain two positions matching with $MMP(0, 2, 0, 0)$.

σ avoiding 132	suffix of σ of length 3	packed of the suffix	$mmp(0, 2, 0, 0)(\sigma)$
1234	234	123	0
2134	134	123	0
2314	314	213	1
2341	341	231	1
3124	124	123	0
3214	214	213	1
3241	241	231	1
3412	412	312	2
3421	421	321	2
4123	123	123	0
4213	213	213	1
4231	231	231	1
4312	312	312	2
4321	321	321	2

3.2. A new bijection between 132-avoiding permutations and non-decreasing parking functions.

Let $\sigma \in \mathfrak{S}_n$ be a 132-avoiding permutation. We set

$$(4) \quad \phi(\sigma) := (mmp(0, n, 0, 0) + 1, mmp(0, n - 1, 0, 0) + 1, \dots, mmp(0, 1, 0, 0) + 1)$$

Theorem 3.3. *Let n be a positive integer. ϕ establishes a bijection between 132-avoiding permutations of length n and non-decreasing parking functions of length n .*

Corollary 3.4. *The number of 132-avoiding permutations containing k values having a greater value on their left is the coefficient $C_{n,k}$ of the Catalan triangle: $Q_{132}^{(0, 1, 0, 0)}|_{t^n x^k} = C_{n,k}$.*

Theorem 3.5. $Q_{n,132}^{(0, k, 0, 0)}(0)$ is equal to the sum of the k first values of the n^{th} row of the Catalan triangle.

From a combinatorial point of view, it is simpler to present $Q_{n,132}^{(0,k,0,0)}(0)$ as a series in t in columns:

$$(5) \quad \begin{array}{c|cccccccc} n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\ 4 & 1 & 4 & 9 & 14 & 14 & 14 & 14 & 14 \\ 5 & 1 & 5 & 14 & 28 & 42 & 42 & 42 & 42 \\ 6 & 1 & 6 & 20 & 48 & 90 & 132 & 132 & 132 \\ 7 & 1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 \\ 8 & 1 & 8 & 35 & 110 & 275 & 572 & 1001 & 1430 \end{array}$$

We thus recognize a simple recurrence

Theorem 3.6. *We have*

$$(6) \quad Q_{n,132}^{(0,k,0,0)}(0) = \begin{cases} 1 & \text{if } n = 1 \text{ or } k = 1, \\ Q_{n-1,132}^{(0,k,0,0)}(0) + Q_{n,132}^{(0,k-1,0,0)}(0) & \text{if } n \geq k, \\ Q_{n,132}^{(0,k-1,0,0)}(0) & \text{if } n < k. \end{cases}$$

We can prove this property with at least three different approaches: the first one is analytic by noticing that Formula (24) presented by [KRT12] at $x = 0$ imply our relations, the second method builds a bijection between the related sets, the last one shows that simple combinatorics objets counted by these numbers satisfy the induction.

4. QUADRANT III: THE SERIES $Q_{132}^{(0,0,\ell,0)}$

These series refine the 132-avoiding permutations along the number of values having ℓ smaller values on their left.

We recall that the Narayana triangle (A001263 of [Slo03])

$$(7) \quad \begin{array}{c|cccccccc} n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & & & & & & & \\ 2 & 1 & 1 & & & & & & \\ 3 & 1 & 3 & 1 & & & & & \\ 4 & 1 & 6 & 6 & 1 & & & & \\ 5 & 1 & 10 & 20 & 10 & 1 & & & \\ 6 & 1 & 15 & 50 & 50 & 15 & 1 & & \\ 7 & 1 & 21 & 105 & 175 & 105 & 21 & 1 & \\ 8 & 1 & 28 & 196 & 490 & 490 & 196 & 28 & 1 \end{array}$$

We will denote $N(n, k)$ the Narayana number indexed by n and k .

Theorem 4.1. *We have $\forall n \geq 1, \forall 0 \leq k \leq n - 1$,*

$$(8) \quad Q_{132}^{(0,0,1,0)}(t, x)|_{t^n x^k} = N(n, k + 1) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$$

This theorem precise Theorem 11 of [KRT12].

A way to see that is to consider the binary decreasing tree associated with a 132-avoiding permutation.

Proposition 4.2. *Let σ be a 132-avoiding permutation of \mathfrak{S} . The number $mmp(0, 0, 1, 0)$ of positions matching with $MMP(0, 0, 1, 0)$ is equal to the number of left branches in the binary decreasing tree associated with σ .*

Proposition 4.3. *The number of binary trees over n nodes containing k left (or right) branches is counted by the Narayana number $N(n, k + 1)$.*

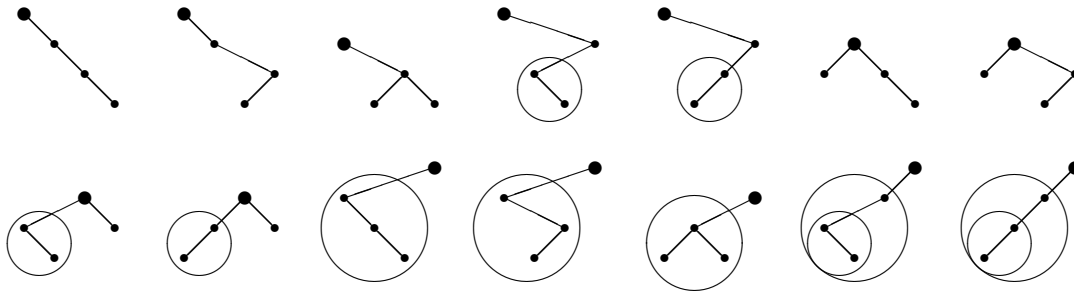


FIGURE 3. Decreasing trees of 132-avoiding permutations of length 4.

Proposition 4.3 and the previous observation conclude Theorem 4.1. We now extend this previous statement from pattern $(0, 0, 1, 0)$ to pattern $(0, 0, \ell, 0)$ with ℓ any positive integer.

Theorem 4.4. $Q_{132}^{(0,0,\ell,0)}|_{t^n x^k}$ is equal to the number of binary trees over n nodes containing themselves k left subtrees containing at least ℓ nodes.

For example, with $\ell = 2$ and $n = 4$, we have $Q_{4,132}^{(0,0,\ell,0)}(x) = 5 + 7x + 2x^2$. As one can check on Figure 3 displaying the fourteen binary trees over 4 nodes in which the left sub-trees over at least 2 nodes have been circled.

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REFERENCES

- [BC11] Petter Brändén and Anders Claesson. Mesh patterns and the expansion of permutation statistics as sums of permutation patterns. *Electron. J. Combin.*, 18(2):Paper 5, 14, 2011.
- [Kra01] C. Krattenthaler. Permutations with restricted patterns and Dyck paths. *Adv. in Appl. Math.*, 27(2-3):510–530, 2001. Special issue in honor of Dominique Foata’s 65th birthday (Philadelphia, PA, 2000).
- [KRT12] Sergey Kitaev, Jeffrey Remmel, and Mark Tiefenbruck. Marked mesh patterns in 132-avoiding permutations i, 2012.
- [S⁺09] W.A. Stein et al. *Sage Mathematics Software (Version 3.3)*. The Sage Development Team, 2009. <http://www.sagemath.org>.
- [SCc08] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008.
- [Slo03] N. J. A. Sloane. The on-line encyclopedia of integer sequences. *Notices Amer. Math. Soc.*, 50(8):912–915, 2003.
- [Úlf11] Henning Úlfarsson. A unification of permutation patterns related to Schubert varieties. *Pure Math. Appl. (P.U.M.A.)*, 22(2):273–296, 2011.

UNIV. PARIS EST MARNE-LA-VALLÉE, LABORATOIRE D’INFORMATIQUE GASPARD MONGE, CITÉ DESCARTES, BÂT COPERNIC – 5, BD DESCARTES CHAMPS SUR MARNE 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE